International Journal of Theoretical Physics, Vol. 46, No. 3, March 2007 (© 2007)
DOI: 10.1007/s10773-005-9100-1

Characteristic Length in a Linear Acceleration

Amos Harpaz[1]

Received April 16, 2005; accepted November 24, 2005
Published Online: February 8, 2007

The role of the characteristic length that characterizes linear acceleration is studied, in order to find how does this length determine the characteristic wavelength of the radiation created by the accelerated charge. Unruh equation for the temperature observed by a detector accelerated relative to the vacuum is used to determine the wavelength distribution of the radiation emitted by a linearly accelerated charge, and it is found that this distribution is peaked close to the characteristic length that characterizes the linear acceleration, which is the radius of curvature of the curved electric field created by the acelerated charge.

KEY WORDS: curved electric field; radius of curvature; radiation wavelength.

PACS numbers: 11.10; 41.60.m.

1. INTRODUCTION

In any process in which radiation is created one should find a characteristic length to be involved, where this characteristic length determines the characteristic wavelength of the radiation created. For example, Singal (1997) argues that a charge accelerated linearly, does not radiate, because no characteristic length is involved in such a motion, and thus no characteristic wavelength can be determined. We shall return latter to this point. Certainly, a characteristic wavelengh can be transformed to a characteristic frequency, by which the radiation frequency is determined. For a radiation created in a periodic motion, like in a cyclotron or in a linear antenna, the situation is very simple—the frequency of the periodic motion determines the characteristic frequency of the radiation (Panofsky and Philips, 1964, p. 252). We shall study here the case of a linear acceleration, in order to define the characteristic length that characterizes this process.

The basic equations for the electric field of a linearly accelerated charge were calculated by Schott (1912) in three dimensional coordinate system, and by Fulton and Rohrlich (1960) in a four dimensional form. The equation of motion for this

[1] Institute of Theoretical Physics, Technion, Haifa, Israel; e-mail: phr89ah@tx.technion.ac.il.

case is given by:

$$z^2 = \alpha^2 + c^2t^2, \tag{1}$$

here α is a constant. The equations for the electric field of this charge are given in cylindrical coordinates (z, ρ, ϕ):

$$E_\rho = \frac{8e\alpha^2\rho z}{\xi^3}, \tag{2}$$

$$E_z = \frac{-4e\alpha^2}{\xi^3}[\alpha^2 + c^2t^2 + \rho^2 - z^2], \tag{3}$$

where $\xi^2 = (\alpha^2 + c^2t^2 - \rho^2 - z^2)^2 + (2\alpha\rho)^2$, a is the charge acceleration, and α (used in equation 1) $\alpha = c^2/a$ is the charge location at $t = 0$ (the turning point of the charge motion in Rindler coordinates (1966)).

The field lines defined by these equations are certainly curved. Singal (1997) calculated from Eqs. (2) and (3) the equation for the field lines, and draw them in a figure. These field lines are displayed in Fig. 1. From the equations calculated by Singal, we calculated the radius of curvature of the electric field lines, R_c, for this case (Harpaz and Soker, 1998), and it is found to be:

$$R_c = \frac{c^2}{a\sin\theta} = \frac{\alpha}{\sin\theta}, \tag{4}$$

where a is the acceleration of the charge, c is the light velocity, and θ is the angle between the particle trajectory and the field line at the point the field "leaves" the charge. Certainly, at this point the velocity of the particle vanishes. From Eq. (4), we learn that in addition to the (geometrical) role of α in Rindler coordinates, this

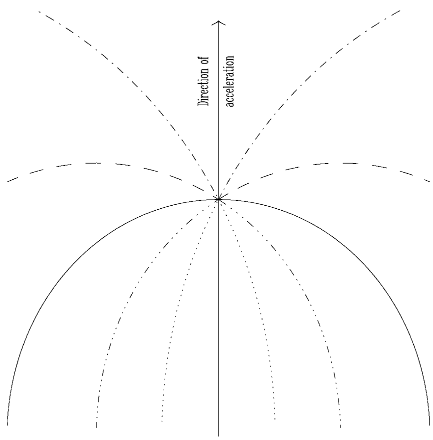

Fig. 1. The field lines of a linearly accelerated charge.

length has a physical meaning, as it characterizes the radius of curvature of the electric field lines, (and thus the "strength" of curvature) where this curvature is created due to the acceleration.

In further investigations of the role of this function, we find that R_c is involved in the determination of the stress force density, f_s, that exists in the curved electric field, given by:

$$f_s = \frac{E^2}{4\pi R_c}, \tag{5}$$

where E is the electric field of the accelerated charge. Later it is found (Harpaz and Soker, 1998, 2001) that this stress force is actually the reaction force, which is responsible for the creation of the electromagnetic radiation. We use Unruh approach (Unruh, 1976) for the temperature observed by a detector accelerated relative to the vacuum, to calculate the characteristic wavelength of the radiation emitted by a linearly accelerated charge (Harpaz and Soker, 2001), and it is found that the characteristic wavelength in this case is few times R_c, the characteristic radius of curvature of the electric field of the accelerated charge.

Thus we find that R_c, the characteristic radius of curvature of the electric field of an accelerated charge has important roles, both geometrical (in Rindler coordinates), and physical (in determining the stress force that creates the radiation), and in characterizing the spectrum of the radiation created in this process. This entity should be studied as a key function that describes the phenomena connected with radiation.

2. THE CREATION OF RADIATION

In order to calculate the reaction force created by the field curvature, we should integrate the stress force density, f_s (Eq. 5) over the space around the charge. We carry this integration in a free system of reference, S, which momentarily coincides with the system of reference of the accelerated charge at time $t = 0$. As we shall see later, the main contribution to the integral comes from the close vicinity of the charge. In this vicinity, the electric field of the charge does not change much from its $1/r^2$ dependence. Hence, we substitute for f_s:

$$f_s = \frac{e^2}{4\pi r^4 R_c} = \frac{e^2\, a \sin\theta}{4\pi c^2 r^4}. \tag{6}$$

The component of the stress force which is parallel to the direction of the acceleration is given by: $-f_s \sin\phi$, where ϕ is the angle between the field line and the direction of the acceleration. Close to the charge $\phi \simeq \theta$, and with this substitution we get:

$$-f_\mathrm{s}\sin\phi \simeq \frac{-e^2\,a\sin^2\theta}{4\pi c^2 r^4}, \tag{7}$$

and in order to find the stress force, F_s , we should integrate this value over the space around the charge. We integrate over $r,\ \theta,\ \phi$. We cannot take the lower limit of r as $r = 0$, because at this limit the integral diverges. To avoid this divergence we take for the lower limit of r, $r = c\Delta t$, where Δt is infinitesimal. As the upper limit for r we take some value r_{up}, where we demand that $c\Delta t \ll r_{\mathrm{up}} \ll c^2/a$. The integral on $d\phi$ yields 2π, and the integral on $\sin^3\theta\, d\theta$ yields $4/3$. Thus we find for the stress force F_s:

$$F_\mathrm{s} = \int -f_\mathrm{s}\sin\phi\, dV = \frac{-2e^2a}{3c^2}\int_{\mathrm{c}\Delta t}^{r_{\mathrm{up}}}\frac{dr}{r^2} = \frac{-2e^2a}{3c^3\Delta t}\left[1 - \frac{c\Delta t}{r_{\mathrm{up}}}\right]. \tag{8}$$

Certainly the second term in the square brackets can be neglected and we are left with:

$$F_\mathrm{s} = \frac{-2e^2a}{3c^3\Delta t}. \tag{9}$$

In order to find the power P, created by the force that overcomes this reaction force, we multiply $-F_\mathrm{s}$ by the velocity of the charge at time $t = \Delta t$, $v = a\Delta t$ and get:

$$P = -F_\mathrm{s}\, a\Delta t = \frac{2e^2a^2}{3c^3}, \tag{10}$$

which is exactly Larmor formula for the power carried by the radiation.

3. THE SPECTRUM OF THE RADIATION

The question what is the frequency of the radiation from a linearly accelerated charge, arises since a charge moving in such a motion has no periodicity in its motion. This led Singal (1997) to argue that a uniformly accelerated charge does not emit radiation at all. Although there is no periodic motion, there is a typical length, which is the characteristic radius of curvature of the electric field, $R_\mathrm{c} = \alpha = c^2/a$, where a is the acceleration. The typical time scale connected with the characteristic length is $t_\mathrm{c} = R_\mathrm{c}/c$,

$$R_\mathrm{c} = c^2/a; \qquad t_\mathrm{c} = c/a. \tag{11}$$

As was shown earlier, the reaction force which drives the creation of the radiation is created by the curvature stress in the curved electric field, and thus, the property that determines the character of the emitted radiation by an accelerated charged particle is the characteristic curvature of the electric field and this sets the length scale and time scale of the problem.

When the radius of curvature does not change with time, or when the time scale of its variation $|a/\dot{a}|$ ($\dot{a} \equiv da/dt$), is much longer than t_c, the field energy of the charged particle reaches thermodynamic equilibrium with the emitted radiation. Hereafter we refer by uniformly accelerated charge to the case where $|a/\dot{a}| \gg t_c$.

If on the other hand $|a/\dot{a}| \ll t_c$, then the time scale of the system will be $t_a = |a/\dot{a}|$, and the typical wave length of the radiation is $ct_a = c|a/\dot{a}|$. For example, in a cyclotron, where charges are moving in rotational motion with a characteristic radius R_s, and angular velocity ω, their radial acceleration is $a = \omega^2 R_s$. The rate of change of their (vector) acceleration is ω. Thus, $t_a = a/\dot{a} = 1/\omega$, while $t_c = \frac{c}{a} = \frac{c}{\omega^2 R_s} = \frac{c}{v}\frac{1}{\omega} = \frac{c}{v} t_a$. Hence, we have $t_a < t_c$, and the spectrum of radiation is determined by the frequency of the system. A similar case is the radiation from a linear antenna, where the frequency of the radiation is determined by the frequency of the alternating currents in the antenna.

In the limit of very long time scales, a thermodynamic equilibrium is reached between the emitted radiation and the energy of the electric field of the accelerated charge. In this case we can define a temperature, T_a, similar to Unruh temperature (Unruh, 1976):

$$kT_a = \frac{\hbar a}{2\pi c}. \tag{12}$$

The blackbody spectrum peaks at

$$\lambda_{max} = \frac{ch}{4.97\,kT} = \frac{(2\pi)^2}{4.97}\frac{c^2}{a} = 7.94 R_c, \tag{13}$$

where we have substituted T_a for the temperature. As expected, for a uniformly accelerated charged particle, the typical wavelength is of the order of magnitude of the radius of curvature of the electric field, c^2/a .

It is interesting to note that similar calculations were used by Bekenstein (9), where he calculates the spectrum of the radiation from an object located very close to the horizon of a black hole, where he uses for the characteristic length the proper distance of the object from the horizon.

From λ_{max} one can calculate ω_{max}, the frequency of the maxima in the radiation curve, to find:

$$\omega_{max} = \frac{2\pi c}{\lambda_{max}} = \frac{2\pi}{7.94}\frac{a}{c}. \tag{14}$$

Similar value was found by Eriksen and Gron (2000) when they performed a Fourier transform of the characteristic length α. We shall return to this topic in the next section.

4. THE CHARACTERISTIC LENGTH

We have learnt in the preceding section that a process in which radiation is created, should involve a characteristic length (or a characteristic frequency) which determines the characteristic wavelength (or the characteristic frequency) of the radiation created in the process. In the case of a periodic acceleration (in a cyclotron or in a linear antenna), the characteristic length of the motion is not constant, but changes periodically with the period of the accelerator. Certainly, in such a case the periodicity of the accelerator will dominate, and it will determine the frequency of the radiation created.

It was already shown (Harpaz and Soker, 2003) that also in the case of a periodic acceleration, the stress of the curved electric field of the accelerated charge can be calculated, according to equations (6–9). In these calculations, the period of the system is included in the calculation of the stress force, through the acceleration involved in equations (6–9), and this period determines the period of the radiation.

However, in a linear acceleration, no periodicity is included in the acceleration, and thus no periodicity is included in the calculation of the stress force. The only length (period) parameter involved in the system is the radius of curvature of the electric field. Eriksen and Gron (2000) performed a Fourier transform to arrive at a characteristic frequency for the radiation. As the characteristic length for this tarnsform, they took α (L in their notation, equation 6.16 in Eriksen and Gron, 2000). The geometric role of this parameter is very clear—it is the closest approach of the trajectory of the accelerated charge to the coordinate origin in Rindler coordinates. The geometric role of this variable is used by Eriksen and Gron. However, when we calculate the curvature of the electric field of the accelerated charge we find that this variable has also a very clear physical role—it characterizes the curvature of the electric field, and this curvature creates the stress force which is the reaction force that drives the creation of the radiation. It is not surprising then, that this length characterizes the wavelength of the radiation created in this process.

In Section 3 we use an entirely different approach—we use Unruh formula for the equilibrium temperature observed by a detector accelerated relative to the vacuum, to calculate the wavelength distribution of the radiation emitted by a linearly accelerated charge. Not surprisingly we find that the wavelength peaks at a wavelength which is few times (about 2π) the characteristic length α. Thus we find that this use gives a physical meaning to the fact that this characteristic length is a key factor in creating the stress force which is the reaction force responsible for the creation of the radiation.

5. CONCLUSIONS

The characteristic wavelength of the radiation created by a linearly accelerated charge is determined by the length that characterizes this motion, which is the characteristic radius of curvature of the curved electric field of the accelerated charge. This curvature creates the reaction force which is responsible for the creation of the radiation. This characteristic length has also a geometrical role, as it is the closest approach of the trajectory of the accelerated charge to the coordinate origin in Rindler coordinates. The characteristic wavelength of the radiation is calculated by an alternative approach, using Unrue formula for the temperature observed by a detector accelerated in vacuum. Using this temperature to calculate the wavelength distribution, it is found that the wavelength peaks at about 2π times this characteristic length, R_c. This actually justifies the use made by Eriksen and Gron, where they performed a Fourier transform of the geometrical meaning of this length to find the characteristic frequency of the radiation created by the accelerated charge.

ACKNOWLEDGMENT

I gratefully acknowledges the help in mathematical treatment, I received from Dr. Ella Shmukler from the Technion.

REFERENCES

Bekenstein, J. D. (1999). *Physical Review* **D60**, 124010.
Eriksen, E. and Gron, O. (2000). *Annals of Physics* **286**, 343.
Fulton, R. and Rohrlich, F. (1960). *Annals of Physics* **9**, 499.
Harpaz, A. and Soker, N. (1998). *General Relativity and Gravitation* **30**, 1217.
Harpaz, A. and Soker, N. (2001). *Foundations of Physics* **31**, 935.
Harpaz, A. and Soker, N. (2003). *Foundations of Physics* **33**, 1207.
Panofsky, W. K. H., and Philips, M. (1964). *Classical Electricity and Magnetism*, Addison-Wesley.
Rindler, W. (1966). *Special Relativity*, Oliver and Boyd.
Schott, G. A. (1912). *Elecromagnetic Radiation*, Cambridge University Press.
Singal, A. K. (1997). *General Relativity Gravitation* **29**, 1371.
Unruh, W. G. (1976). *Physical Review* **D14**, 169.